Quantum synchronization and correlations of two mechanical resonators in a dissipative optomechanical system

Chang-Geng Liao,[1,2,3] Rong-Xin Chen,[4,*] Hong Xie,[5] Meng-Ying He,[1,2] and Xiu-Min Lin[1,2,†]

[1]*Fujian Provincial Key Laboratory of Quantum Manipulation and New Energy Materials, College of Physics and Energy, Fujian Normal University, Fuzhou 350117, China*

[2]*Fujian Provincial Collaborative Innovation Center for Optoelectronic Semiconductors and Efficient Devices, Xiamen 361005, China*

[3]*Department of Electronic Engineering, Fujian Polytechnic of Information Technology, Fuzhou 350003, China*

[4]*School of Mechanical & Electrical Engineering, Longyan University, Longyan 364012, China*

[5]*Department of Mathematics and Physics, Fujian Jiangxia University, Fuzhou 350108, China*

(Received 20 December 2018; published 11 March 2019)

Quantum synchronization and its connection with other quantum correlations have attracted considerable attention. Here we present a theoretical scheme to simultaneously represent and significantly enhance the level of quantum synchronization and entanglement between two indirectly coupled mechanical membranes, which are coupled to a common optical field within a cavity. By applying a two-tone driving laser with weighted amplitudes and specific frequencies, both synchronization gauged by Mari's criterion and entanglement estimated by logarithmic negativity can be greatly enhanced. We then clarify the relationship between quantum synchronization and entanglement in detail. Numerical simulation results show that the influence of the coupling asymmetry G_2/G_1 on quantum complete synchronization behaves similarly to that on the purity while the influence of the coupling asymmetry G_2/G_1 on quantum phase synchronization is more similar to that on quantum entanglement. Besides, we demonstrate that although quantum synchronization and quantum entanglement are not directly related, both of them are sensitive to the squeezing parameter and the cooling effect. Furthermore, it is also shown that detuning the frequencies of two mechanical oscillators can actually help quantum synchronization and entanglement, which is somewhat similar to the case of quantum synchronization blockade.

DOI: 10.1103/PhysRevA.99.033818

I. INTRODUCTION

Synchronization phenomena are universal and can be found extensively in science, nature, engineering, and social life [1]. Since Huygens first observed the unique phenomenon of synchronization between two pendulum clocks [2], a large variety of different contexts about synchronization have been studied. Nevertheless, only in recent years have people begun to explore synchronization of quantum systems in the quantum regime. These systems involve cavity quantum electrodynamics [3,4], atomic ensembles [5–9], van der Pol oscillators [10–14], Josephson junction [15,16], and so on. Among them, the optomechanical system provides an ideal platform for investigating spontaneous synchronization [17–26] due to its inherent nonlinear nature. Remarkably, phase synchronization of two anharmonic nanomechanical oscillators had been experimentally implemented and a significant reduction in the phase noise of oscillators in the synchronized state had been demonstrated [17], which is key for sensor and clock applications.

Classical synchronization techniques have several inimitable applications like encrypted communications [27], frequency stabilization of the powerful generator [28], etc. Similar to the classic case, the synchronization of quantum systems in the quantum regime can also serve as a tool in many applications. For example, the spectral density of a dissipative qubit can be probed via quantum synchronization [29]. However, the laws hidden in quantum synchronization are quite different from those in classical synchronization, which further arouses intense research interest. Noise-induced classical-to-quantum transitions in optomechanical synchronization have been investigated [30]. Besides, it is counterintuitively shown that identical self-oscillators in the deep quantum regime cannot synchronize, which is called quantum synchronization blockade and can be observed in circuit quantum electrodynamics [16,31], and detuning their frequencies can actually help synchronization [31]. Notably, as a kind of temporal correlation between subsystems, synchronization is also related to other correlations such as mutual information, discord, and entanglement [32]. The common ingredient for the emergence of these features is mutual interaction between subsystems. Intriguingly, it is demonstrated that classical synchronization indicates persistent entanglement in isolated quantum systems [33]. In Ref. [34], the authors identified the conditions leading to the phenomenon of mutual synchronization, showing that the ability of the system to synchronize is related to the existence of disparate decay rates and accompanied by robust quantum discord and mutual information

*chenrxas@163.com

†xmlin@fjnu.edu.cn

between oscillators. Another work [35] showed that while both the quantum mutual information and the presence of entanglement appear to reproduce the Arnold tongue, they do not provide a conclusive measure of synchronization.

The advantages of the optomechanical system, such as having a wide range of parameters and enabling presentation of phenomena of both classical and quantum systems [36], promote the development of entanglement and synchronization. Many theoretical schemes [37–53] have shown that entanglement can be enhanced by suitably modulating a driving laser or combining with a dissipative regime. Very recently, entanglement and nonclassical correlations [54,55] between remote mechanical systems comprised of billions of atoms have been experimentally realized. Hence, people naturally think about whether the modulation and dissipative regime can be used to enhance the level of quantum synchronization or other types of quantum correlations. Preliminary explorations indicate that quantum synchronization can indeed be enhanced by a proper periodic modification [56,57]. Besides, quantum synchronization can also be enhanced by driving a self-sustained oscillator with a squeezing Hamiltonian instead of a harmonic drive [58]. However, it is still difficult to grasp the essence of these correlations.

In this paper, we further clarify the relationship between quantum synchronization and entanglement in a dissipative optomechanical system. By applying a two-tone driving laser with weighted amplitudes and specific frequencies, both the level of synchronization measured by Mari's criterion and entanglement estimated by logarithmic negativity can be represented simultaneously and greatly enhanced. Although the stationary state of two mechanical resonators can possess a maximum amount of complete or phase synchronization without being necessarily entangled, which has been pointed out by previous studies [22,24], there is a certain interplay between synchronization and entanglement. Based on these consequences, a richer connection between quantum synchronization and quantum entanglement can be obtained. Numerical simulation results show that the curve of quantum complete synchronization with the coupling asymmetry G_2/G_1 as a variable is similar to that of the purity while the variation trends with the coupling asymmetry G_2/G_1 of quantum phase synchronization and entanglement are alike. Besides, we demonstrate that although quantum synchronization and quantum entanglement are not directly related, both of them are sensitive to the squeezing parameter and the cooling effect. In other words, both squeezing and cooling effects can enhance quantum complete synchronization and quantum phase synchronization as well as quantum entanglement. Furthermore, detuning the frequencies of two mechanical oscillators can actually help synchronization and entanglement, which is somewhat similar to the situation of quantum synchronization blockade [16,31].

II. THEORETICAL MODEL

The system is depicted in Fig. 1 where two dielectric membranes acting as two mechanical oscillators are placed within an optical cavity driven by a laser with frequency ω_L and time-modulated amplitude $E(t)$. The Hamiltonian (in a frame rotating with the laser frequency ω_L) of the system can be written as follows ($\hbar = 1$):

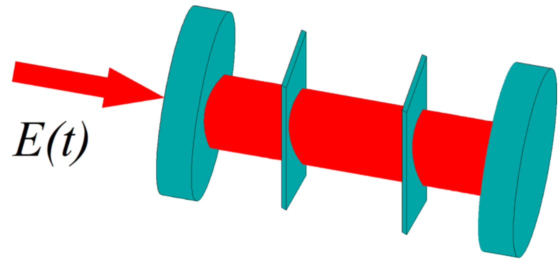

FIG. 1. Schematic representation of the system. Two dielectric membranes acting as two mechanical oscillators are placed within an optical cavity, which is driven by an amplitude-modulated laser $E(t)$.

$$H = \Delta A^\dagger A + \sum_{j=1,2}\left[\frac{\Omega_j}{2}\left(P_j^2 + Q_j^2\right) + gA^\dagger A Q_j\right] + iE(t)A^\dagger - iE^*(t)A. \quad (1)$$

Here, $\Delta = \omega_c - \omega_L$ denotes the cavity mode detuning, $A^\dagger$ (A) refers to the creation (annihilation) operator of the cavity mode with frequency ω_c and decay rate κ, and Q_j (P_j) represents the dimensionless position (momentum) operator of the jth mechanical oscillator with frequency Ω_j and damping rate γ_j. The optomechanical coupling constant g is assumed to be real and the same for both mechanical oscillators for simplicity.

The dissipative dynamics of the system can be described by a set of nonlinear quantum Langevin equations (QLEs):

$$\dot{A} = -(\kappa + i\Delta)A - igA(Q_1 + Q_2) + E(t) + \sqrt{2\kappa}a^{\text{in}}(t), \quad (2a)$$

$$\dot{Q}_j = \Omega_j P_j, \quad (2b)$$

$$\dot{P}_j = -\Omega_j Q_j - gA^\dagger A - \gamma_j P_j + \xi_j(t), \quad (2c)$$

where $a^{\text{in}}(t)$ is the vacuum input noise operator, with the only nonzero correlation function [59]

$$\langle a^{\text{in}}(t)a^{\text{in}\dagger}(t')\rangle = \delta(t - t'). \quad (3)$$

The correlation function of the zero-mean Brownian motion noise operator $\xi_j(t)$ in the case of the large mechanical quality $\mathbf{Q}_j = \Omega_j/\gamma_j \gg 1$ can be approximately described by the Markovian process and satisfies [60]

$$\langle \xi_j(t)\xi_j(t') + \xi_j(t')\xi_j(t)\rangle/2 = \gamma_j(2\bar{n}_{\text{th}} + 1)\delta(t - t'), \quad (4)$$

where $\bar{n}_{\text{th}} = [\exp(\hbar\Omega_j/k_B T) - 1]^{-1}$ is the mean thermal phonon number at the environmental temperature T.

When the system is strongly driven to a large classical mean value, the standard linearization technique can be adopted and each Heisenberg operator can be properly described as $O = \langle O(t)\rangle + o(t)$ [$O(o) = Q_j(q_j), P_j(p_j), A(a)$], where $\langle O(t)\rangle$ represents the classical c-number mean value and $o(t)$ denotes quantum fluctuation around the classical mean value. By substituting $O = \langle O(t)\rangle + o(t)$ into Eq. (2), we obtain the following differential equations for the classical mean values:

$$\langle \dot{A} \rangle = -(\kappa + i\Delta)\langle A\rangle - ig\langle A\rangle(\langle Q_1\rangle + \langle Q_2\rangle) + E(t), \quad (5a)$$

$$\langle \dot{Q}_j\rangle = \Omega_j\langle P_j\rangle, \quad (5b)$$

$$\langle \dot{P}_j\rangle = -\Omega_j\langle Q_j\rangle - \gamma_j\langle P_j\rangle - g\langle A\rangle^*\langle A\rangle. \quad (5c)$$

The linearized QLEs for the quantum fluctuations are

$$\dot{a} = -(\kappa + i\Delta)a - ig[\langle A\rangle(q_1 + q_2) + (\langle Q_1\rangle + \langle Q_2\rangle)a] + \sqrt{2\kappa}a^{\text{in}}(t), \quad (6a)$$

$$\dot{q}_j = \Omega_j p_j, \quad (6b)$$

$$\dot{p}_j = -\Omega_j q_j - \gamma_j p_j - g(\langle A\rangle^* a + \langle A\rangle a^\dagger) + \xi_j(t), \quad (6c)$$

and the corresponding linearized system Hamiltonian

$$H^{\text{lin}} = \Delta(t)a^\dagger a + \sum_{j=1,2}\left\{\frac{\Omega_j}{2}\left(p_j^2 + q_j^2\right) + [G^*(t)a + G(t)a^\dagger]q_j\right\}, \quad (7)$$

with the effective detuning $\Delta(t) = \Delta + g(\langle Q_1\rangle + \langle Q_2\rangle)$ and effective coupling strength $G(t) = g\langle A\rangle$.

III. EFFECTIVE HAMILTONIAN

It is difficult to find an exact solution of the nonlinear differential Eq. (5) in general. Here we focus on the weak optomechanical coupling regime, i.e., $|g/\Omega_j| \ll 1$. It is sufficient to only consider the zero-order term of g for the classical mean values. In the long-time limit, we have

$$\langle A(t)\rangle^{(0)} = \sum_{l=1,2}\frac{E_l}{\kappa - i(\omega_l - \Delta)}e^{-i\omega_l t}, \quad (8a)$$

$$\langle P_1\rangle^{(0)} = \langle P_2\rangle^{(0)} = 0, \quad (8b)$$

$$\langle Q_1\rangle^{(0)} = \langle Q_2\rangle^{(0)} = 0, \quad (8c)$$

by applying a two-tone driving laser with weighted amplitudes and specific frequencies $E(t) = \sum_{l=1}^2 E_l e^{-iw_l t}$. By introducing the creation and annihilation operators of the mechanical fluctuation

$$b_j = (q_j + ip_j)/\sqrt{2}, \quad b_j^\dagger = (q_j - ip_j)/\sqrt{2}, \quad (9)$$

Eq. (7) can be rewritten as

$$H^{\text{lin}} \simeq \Delta a^\dagger a + \sum_{j=1,2}\{\Omega_j b_j^\dagger b_j + [\widetilde{G}^*(t)a + \widetilde{G}(t)a^\dagger](b_j + b_j^\dagger)\}, \quad (10)$$

where the effective coupling of the zero-order approximation is

$$\widetilde{G}(t) = \sum_{l=1,2} G_l e^{-i\omega_l t} \quad (11)$$

with $G_l = gE_l/\{\sqrt{2}[\kappa - i(\omega_l - \Delta)]\}$ (assumed to be real in the following). In the interaction picture of $\Delta a^\dagger a + (\Omega_1 - \delta)b_1^\dagger b_1 + (\Omega_2 + \delta)b_2^\dagger b_2$, Eq. (10) is transformed into

$$H^{\text{lin}} = \delta(b_1^\dagger b_1 - b_2^\dagger b_2) + [\widetilde{G}^*(t)ae^{-i\Delta t} + \widetilde{G}(t)a^\dagger e^{i\Delta t}] \times [b_1 e^{-i(\Omega_1-\delta)t} + b_2 e^{-i(\Omega_2+\delta)t} + \text{H.c.}]. \quad (12)$$

If we set $\delta = (\Omega_1 - \Omega_2)/2$, $w_1 = \Delta - (\Omega_1 + \Omega_2)/2$, and $w_2 = \Delta + (\Omega_1 + \Omega_2)/2$, one can find that H^{lin} is composed of resonant and nonresonant interacting terms

$$H^{\text{lin}} = \delta(b_1^\dagger b_1 - b_2^\dagger b_2) + [G_1 a(b_1^\dagger + b_2^\dagger) + G_2 a(b_1 + b_2) + \text{H.c.}] + [G_1 a(b_1 + b_2)e^{-i(\Omega_1+\Omega_2)t} + G_2 a(b_1^\dagger + b_2^\dagger)e^{i(\Omega_1+\Omega_2)t} + \text{H.c.}]. \quad (13)$$

When the condition $\Omega_1 + \Omega_2 \gg G_1, G_2$ is satisfied, we can neglect the fast oscillating terms under the rotating-wave approximation to get an effective Hamiltonian

$$H^{\text{lin}} = \delta(\beta_1^\dagger\beta_1 - \beta_2^\dagger\beta_2) + [\chi a(\beta_1^\dagger + \beta_2^\dagger) + \text{H.c.}], \quad (14)$$

where the Bogoliubov modes β_1 and β_2 are, separately, unitary transformations of b_1 and b_2 with a two-mode squeezing operator $s(r)$, i.e.,

$$\beta_1 = s(r)b_1 s^\dagger(r) = b_1\cosh r + b_2^\dagger\sinh r, \quad (15a)$$

$$\beta_2 = s(r)b_2 s^\dagger(r) = b_2\cosh r + b_1^\dagger\sinh r, \quad (15b)$$

$\chi = \sqrt{G_1^2 - G_2^2}$ represents the effective coupling between the Bogoliubov mode and the cavity mode (we have assumed $G_2 < G_1$ to ensure stability), and $s(r) = \exp[r(b_1 b_2 - b_1^\dagger b_2^\dagger)]$ with the squeezing parameter $r = \tanh^{-1}(G_2/G_1)$. In terms of the sum mode and the difference mode of Bogoliubov modes

$$\beta_{\text{sum}} = (\beta_1 + \beta_2)/\sqrt{2}, \quad \beta_{\text{diff}} = (\beta_1 - \beta_2)/\sqrt{2}, \quad (16)$$

Eq. (14) becomes

$$H^{\text{lin}} = \delta\beta_{\text{sum}}^\dagger\beta_{\text{diff}} + \sqrt{2}\chi\beta_{\text{sum}}^\dagger a + \text{H.c.} \quad (17)$$

As shown in Refs. [41,52,55], the dissipative dynamics of the cavity mode can be used to cool the Bogoliubov modes towards their joint ground state, which is in fact a two-mode squeezed state of the mechanical oscillators.

IV. QUANTUM SYNCHRONIZATION AND CORRELATIONS

The quantum statistical properties of the system can be obtained through the fluctuations of the operators around the mean values. The fact that the dynamics of the system is governed by a linearized Hamiltonian ensures that the evolved states are Gaussian states, whose properties are fully represented by a 6×6 covariance matrix (CM) σ with components defined as [61]

$$\sigma_{m,n} = \langle R_m R_n + R_n R_m\rangle/2. \quad (18)$$

Here R_n is the nth entry of the vector of quadratures R defined by

$$R = [q_a, p_a, q_1, p_1, q_2, p_2]^T, \quad (19)$$

and the position and momentum quadratures of the bosonic modes ($h \in \{a, a^{\text{in}}\}$) are

$$q_h = (h + h^\dagger)/\sqrt{2}, \quad p_h = (h - h^\dagger)/(i\sqrt{2}). \quad (20)$$

By further introducing the column vector of input noises

$$N(t) = (\sqrt{2\kappa}q_{a^{\text{in}}}(t), \sqrt{2\kappa}p_{a^{\text{in}}}(t), 0, \xi_1(t), 0, \xi_2(t))^{\text{T}}, \quad (21)$$

the corresponding dynamical linearized QLEs in Eq. (6) can be expressed in a compact matrix form as

$$\dot{R} = M(t)R + N(t) \tag{22}$$

with

$$M(t) = \begin{pmatrix} -\kappa & \Delta(t) & \sqrt{2}G_{\rm I}(t) & 0 & \sqrt{2}G_{\rm I}(t) & 0 \\ -\Delta(t) & -\kappa & -\sqrt{2}G_{\rm R}(t) & 0 & -\sqrt{2}G_{\rm R}(t) & 0 \\ 0 & 0 & 0 & \Omega_1 & 0 & 0 \\ -\sqrt{2}G_{\rm R}(t) & -\sqrt{2}G_{\rm I}(t) & -\Omega_1 & -\gamma_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Omega_2 \\ -\sqrt{2}G_{\rm R}(t) & -\sqrt{2}G_{\rm I}(t) & 0 & 0 & -\Omega_2 & -\gamma_2 \end{pmatrix}, \tag{23}$$

where $G_{\rm R}$ and $G_{\rm I}$ are respectively real and imaginary parts of the effective coupling coefficient $G(t)$ in Eq. (7). Since the system is linearized, the evolved states will remain Gaussian and the QLEs in Eq. (6) are equivalent to the equation of motion for the CM. From Eqs. (3), (4), (18), and (22), we can derive a linear differential equation for the CM [37],

$$\dot{\sigma}(t) = M(t)\sigma(t) + \sigma(t)M(t)^{\rm T} + D, \tag{24}$$

where D is a diffusion matrix whose components are associated with the noise correlation functions and defined as

$$\delta(t-t')D_{m,l} = \langle N_m(t)N_l^\dagger(t') + N_l^\dagger(t')N_m(t)\rangle 2. \tag{25}$$

Here N_l is the lth entry of the vector N. One can deduce from Eqs. (3) and (4)

$$D = {\rm diag}(\kappa, \kappa, 0, \gamma_1(2\bar{n}_{\rm th}+1), 0, \gamma_2(2\bar{n}_{\rm th}+1)). \tag{26}$$

Note that the general stability conditions of the linear differential Eqs. (22) or (24) are determined by the corresponding homogeneous equation

$$\dot{R} = M(t)R, \tag{27}$$

which is fully characterized by the time-periodic coefficient matrix $M(t)$. Based on Floquet's theorem [62], the solutions of Eq. (22) or Eq. (24) are stable if all Floquet multipliers satisfy $|\lambda_j| < 1$. Here, λ_j is the jth eigenvalue of $\Lambda = \Pi^{-1}(0)\Pi(T)$ and $\Pi(t)$ is a principal matrix solution of Eq. (27). For the special case of a time-independent coefficient matrix $M = M(t=0)$ under the rotating-wave approximation, i.e., omitting all nonresonant terms in Eq. (13) or all time-dependent terms in Eq. (23), the stability requirements can be readily inferred from the eigenvalues of the time-independent coefficient matrix M, where the system is stable provided that all eigenvalues of M have negative real parts [63]. In the following, the stability conditions will be carefully checked in all simulations throughout this paper.

For two-mode Gaussian states of two mechanical resonators b_1 and b_2, it is convenient to use Mari's criterion [22] as a measurement of the quantum synchronization. As for quantum entanglement, we use the logarithmic negativity $E_{\rm N}$ [64,65] to gauge its level. All the above measures can be computed from the reduced 4×4 CM $\sigma_{\rm r}(t)$ for b_1 and b_2,

$$\sigma_{\rm r}(t) = \begin{pmatrix} \sigma_1 & \sigma_{\rm c} \\ \sigma_{\rm c}^{\rm T} & \sigma_2 \end{pmatrix}, \tag{28}$$

where σ_1, σ_2, and $\sigma_{\rm c}$ are 2×2 sub-block matrices of $\sigma_{\rm r}(t)$. The entanglement is then calculated by

$$E_{\rm N} = \max[0, -\ln(2\eta)] \tag{29}$$

with

$$\eta \equiv 2^{-1/2}\{\Sigma_- - [\Sigma_-^2 - 4I_4]^{1/2}\}^{1/2} \tag{30}$$

and

$$\Sigma_- \equiv I_1 + I_2 - 2I_3, \tag{31}$$

where $I_1 = \det\sigma_1$, $I_2 = \det\sigma_2$, $I_3 = \det\sigma_{\rm c}$, and $I_4 = \det\sigma_{\rm r}$ are symplectic invariants. The purity of a two-mode Gaussian state is simply given by

$$\mu = 1/(4\sqrt{I_4}). \tag{32}$$

As proposed by Mari *et al.* [22], the relative measure of quantum complete synchronization can be expressed as

$$S_{\rm c}(t) = \langle q_-(t)^2 + p_-(t)^2\rangle^{-1}, \tag{33}$$

where

$$q_-(t) = [q_1(t) - q_2(t)]/\sqrt{2}, \tag{34a}$$

$$p_-(t) = [p_1(t) - p_2(t)]/\sqrt{2}. \tag{34b}$$

The measure of quantum phase synchronization can be obtained through the quantity

$$S_{\rm p}(t) = \tfrac{1}{2}\langle p'_-(t)^2\rangle^{-1} \tag{35}$$

with

$$p'_-(t) = [p'_1(t) - p'_2(t)]/\sqrt{2} \tag{36}$$

and

$$q'_j(t) = q_j\cos\varphi_j + p_j\sin\varphi_j, \tag{37a}$$

$$p'_j(t) = p_j\cos\varphi_j - q_j\sin\varphi_j, \tag{37b}$$

where the phase $\varphi_j = \arctan[\langle P_j(t)\rangle/\langle Q_j(t)\rangle]$.

V. NUMERICAL RESULTS AND DISCUSSION

Figure 2 displays the peak values of the mechanical correlations and purity for each time period in the long-time limit as functions of the coupling asymmetry G_2/G_1 for different δ. All numerical results are obtained by solving the full linearized QLEs for the quantum fluctuations Eq. (6) [or equally the linear differential Eq. (24) for the CM] including the nonresonant terms. To do this, we need to first

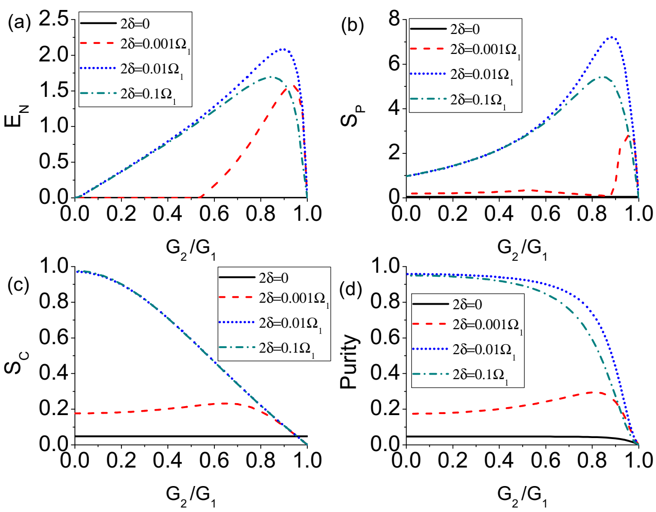

FIG. 2. Maximum mechanical correlations and purity for each time period in the asymptotic regime as functions of the coupling asymmetry G_2/G_1 for different δ. The parameters are $\kappa/\Omega_1 = 0.05$, $\gamma/\Omega_1 = 5\times10^{-6}$, $\Delta/\Omega_1 = -1$, $g/\Omega_1 = 1\times10^{-5}$, $G_1/\Omega_1 = 0.03$, and $\bar{n}_{\rm th} = 10$.

numerically integrate the differential Eq. (5) by applying a two-tone driving laser with weighted amplitudes and specific frequencies $E(t) = \sum_{l=1}^{2} E_l e^{-iw_l t}$ to get the time-dependent mean values for Eqs. (6) and (7). Apparently, the mechanical correlations are nonmonotonic functions of G_2/G_1 in most sets of parameters and take a maximum for a specific G_2/G_1. The increase of the coupling asymmetry G_2/G_1 has two competing effects. On the one hand, it can increase the squeezing parameter $r = \tanh^{-1}(G_2/G_1)$ of the two-mode squeezed thermal state. On the other hand, it will reduce the effective coupling $\chi = \sqrt{G_1^2 - G_2^2}$ between the sum mode $\beta_{\rm sum}$ and the cavity mode a, which is harmful for the cooling effect. As is well known, both the squeezing parameter and the cooling effect play an important role in the generation of various kinds of quantum correlations, but their influences on diverse kinds of quantum correlations are different. From Fig. 2, one can find that the achievable quantum correlations are also dependent on δ, which is the effective coupling between the sum mode $\beta_{\rm sum}$ and the difference mode $\beta_{\rm diff}$ and induces the cooling process of $\beta_{\rm diff}$. The combined cooling effect has been previously studied in Refs. [41,52,55] and used to generate highly pure and strong cavity-mechanical entanglement [52]. By optimizing the coupling asymmetry G_2/G_1 and the effective coupling δ, the first-rank quantum correlations can be obtained. The tendency towards spontaneous synchronization or entanglement is the weakest if the natural frequencies of the two mechanical oscillators are as close as possible (corresponding to $\delta = 0$), which is contrary to the classic case. The result is similar to that of the quantum synchronization blockade [16,31], in which identical self-oscillators cannot synchronize and detuning their frequencies is conducive actually to synchronization. However, the fundamental physical mechanisms of these two cases are different. The appearance of the quantum synchronization blockade is due to energy quantization and energy conservation while the reason for our scheme is that the difference mode $\beta_{\rm diff}$ cannot be effectively cooled in the case of $\delta = 0$. Besides, all other measures of quantum correlation except for quantum entanglement are not exactly zero when the natural frequencies of the two mechanical oscillators are identical.

Noticeably, by comparing Fig. 2(a) with Fig. 2(b), one can find that the curves of $E_{\rm N}$ are very similar to that of quantum phase synchronization $S_{\rm p}$, especially when the parameter δ is large enough. In this case, they have similar dependency on the squeezing parameter and the cooling effect. If δ is very small but not exactly zero (e.g., $\delta = 0.0005\Omega_1$), the tendency of quantum phase synchronization $S_{\rm p}$ becomes more complex. In addition, the peak of phase synchronization $S_{\rm p}$ occurs in a larger ratio of G_2/G_1. As was already pointed out in Ref. [52], for different sets of parameters G_2 and G_1, one expects some moderate values of δ that correspond to maximum entanglement and purity. That is to say, the smaller the parameter δ is, the smaller the optimal coupling $\chi = \sqrt{G_1^2 - G_2^2}$ (i.e., the larger the ratio G_2/G_1) is. This conclusion is also suitable to quantum phase synchronization. In fact, apart from the competition between squeezing and the combined cooling effect of the sum mode $\beta_{\rm sum}$ and the difference mode $\beta_{\rm diff}$, the competing of the cooling effect between two Bogoliubov modes also exists. The best value is obtained when all the competing effects balance.

Figure 2(c) shows that both curves of quantum complete synchronization for $\delta = 0.05\Omega_1$ and $\delta = 0.005\Omega_1$ almost overlap and drop rapidly as the ratio G_2/G_1 grows, which means that both the sum mode $\beta_{\rm sum}$ and the difference mode $\beta_{\rm diff}$ can be effectively cooled via the dissipative dynamics of the cavity mode a under the conditions of large δ and small G_2/G_1. In other words, the impact of the cooling effect on quantum complete synchronization is much larger than that of squeezing. Therefore, the enhancement effect of quantum complete synchronization induced by squeezing becomes obvious only when the cooling effect is weak enough.

Figure 2(d) reveals that the purity for large δ is inversely correlated to G_2/G_1 and the curves for small δ are very similar to that of quantum complete synchronization. Clearly, under the circumstance of large δ, one can keep the high purity of steady states over a wide range of G_2/G_1, which means the combined cooling effect on the sum mode $\beta_{\rm sum}$ and the difference mode $\beta_{\rm diff}$ can tremendously maintain large purity even if the mean thermal phonon number is not small. If δ is very small, i.e., the coupling between the sum mode $\beta_{\rm sum}$ and the difference mode $\beta_{\rm diff}$ is small, the enlargement of the ratio G_2/G_1 can only increase the squeezing parameter $r = \tanh^{-1}(G_2/G_1)$ but decreases the coupling between the sum mode $\beta_{\rm sum}$ and the cavity mode a. As long as the cooling effect on the sum mode $\beta_{\rm sum}$ is not very small, enlargement of the squeezing parameter can increase the purity. However, the sum mode $\beta_{\rm sum}$ cannot be effectively cooled when the ratio G_2/G_1 is large enough. Hence, the purity falls off due to the sensitivity to the environmental temperature. The same principle can explain the result of quantum complete synchronization for small δ.

To gain more insights about the dynamics, we respectively plot the time evolutions of different mechanical quantum correlations and purity in Fig. 3 when all mechanical and cavity modes are initially in thermal equilibrium with their baths. The results are numerically evaluated with the full

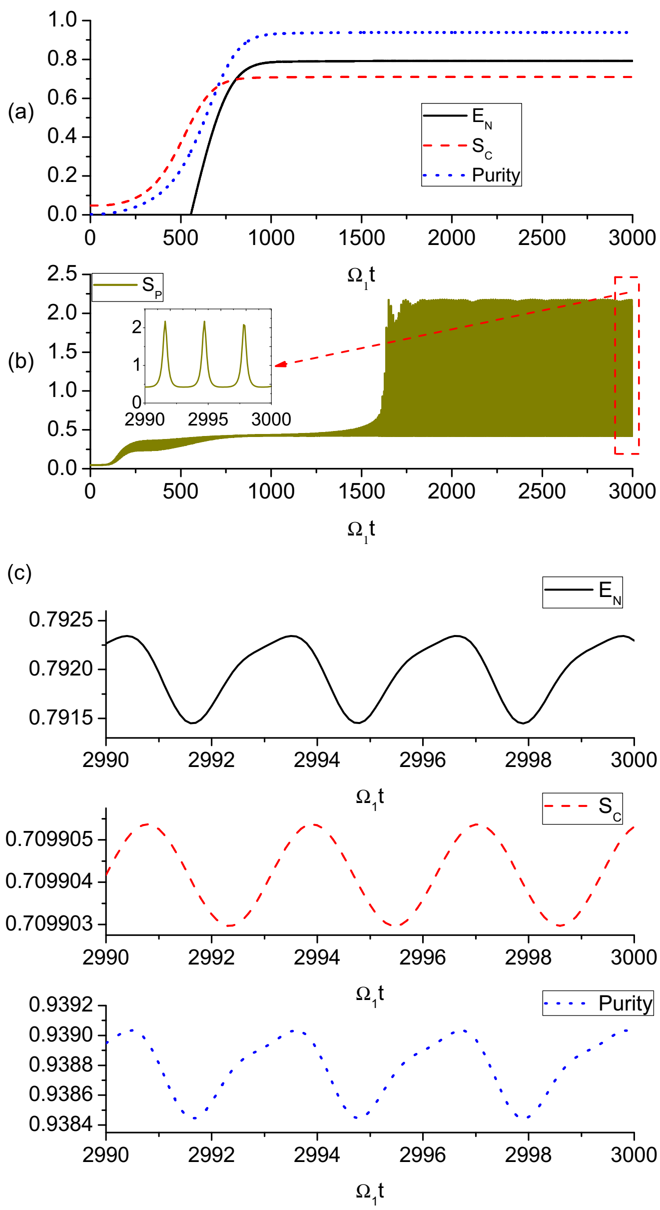

FIG. 3. Time evolutions of different mechanical correlations and purity. The parameters are $\kappa/\Omega_1 = 0.05$, $\gamma/\Omega_1 = 5 \times 10^{-6}$, $\Delta/\Omega_1 = -1$, $g/\Omega_1 = 1 \times 10^{-5}$, $G_1/\Omega_1 = 0.03$, $G_2 = 0.4G_1$, $\delta/\Omega_1 = 0.005$, and $\bar{n}_{\rm th} = 10$.

linearized Hamiltonian in Eq. (7). Figure 3(a) shows that there is no entanglement until the Bogoliubov modes β_1 and β_2 have been sufficiently cooled after some time. Nevertheless, the quantum state of two mechanical resonators can possess a certain amount of quantum correlations (e.g., quantum complete synchronization) without being necessarily entangled. Then, following a dramatic increase, all measures of quantum correlation and purity tend to be saturated with small vibrations [see Fig. 3(c)] which derive from the effects of nonresonant terms. Comparing Figs. 3(a) and 3(b), one notes that the dynamics of quantum phase synchronization is quite different from that of other quantum correlation. After the first saturation with small vibrations [roughly speaking $\Omega_1 t \in (250, 1500)$], the measure of quantum phase synchronization dramatically increases again and then becomes saturation with large vibrations finally [see the inset of Fig. 3(b)]. Although they have different dynamical evolution, all measures of quantum correlation will reach a relatively large value and oscillate with the same period in the long-time limit. As a consequence, the level of all kinds of quantum correlation can be significantly enhanced and simultaneously represented, which provides richer phenomena and better understanding on the relationship among these different quantum correlations.

VI. CONCLUSIONS

In summary, we have systematically explored quantum synchronization and its connection with other quantum correlations in a dissipative three-mode optomechanical system. The results show that combinations of the modulation and the dissipation regime can significantly enhance the level of several different kinds of quantum correlations (including quantum synchronization) between two indirectly coupled mechanical oscillators. Based on these consequences, we clarify the relationship between quantum synchronization and entanglement. Then, we demonstrate that the influence of the coupling asymmetry G_2/G_1 on quantum complete synchronization behaves similarly to that on the purity while the influence of the coupling asymmetry G_2/G_1 on quantum phase synchronization is more similar to that on quantum entanglement. Generally, synchronization is certainly associated with entanglement although complete or phase synchronization can exist without entanglement when the Bogoliubov modes β_1 and β_2 have not been sufficiently cooled. Besides, the tendency towards spontaneous synchronization or entanglement is the weakest when the natural frequencies of two mechanical oscillators are as close as possible. Numerical simulation results show that both squeezing and the combined cooling effects can enhance quantum complete synchronization and quantum phase synchronization as well as quantum entanglement. However, they have different impressions on diverse kinds of quantum correlations. The underlying physical mechanism is that there are two different competing relationships: one is the competition between squeezing and the combined cooling effect of the sum mode $\beta_{\rm sum}$ and the difference mode $\beta_{\rm diff}$; the other is the competing of the cooling effect between two Bogoliubov modes. The best value is obtained when all the competing effects balance. Therefore, our study provides a better understanding on the relation among these different quantum correlations.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China under Grants No. 61275215 and No. 11674059, the Special Foundation for Theoretical Physics Research Program of China under Grant No. 11847056, the Natural Science Foundation of Fujian Province of China under Grants No. 2016J01009 and No. 2016J01008, the Educational Committee of Fujian Province of China under Grants No. JAT160687 and No. JA14397, the 2016 Annual College Funds for Distinguished Young Scientists of Fujian Province of China, and funds from Fujian Polytechnic of Information Technology under Grant No. Y17104.

[1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, UK, 2003).
[2] C. Huygens, *O Euvres complètes de Christiaan Huygens* (Martinus Nijhoff, The Hague, 1893).
[3] V. Ameri, M. Eghbali-Arani, A. Mari, A. Farace, F. Kheirandish, V. Giovannetti, and R. Fazio, Phys. Rev. A **91**, 012301 (2015).
[4] O. V. Zhirov and D. L. Shepelyansky, Phys. Rev. B **80**, 014519 (2009).
[5] M. Xu and M. J. Holland, Phys. Rev. Lett. **114**, 103601 (2015).
[6] M. R. Hush, W. Li, S. Genway, I. Lesanovsky, and A. D. Armour, Phys. Rev. A **91**, 061401 (2015).
[7] M. Xu, D. A. Tieri, E. C. Fine, J. K. Thompson, and M. J. Holland, Phys. Rev. Lett. **113**, 154101 (2014).
[8] H. Qiu, R. Zambrini, A. Polls, J. Martorell, and B. Juliá-Díaz, Phys. Rev. A **92**, 043619 (2015).
[9] L. Zhang, X.-T. Xu, X. Liu, and W. Zhang, arXiv:1703.03704.
[10] S. Walter, A. Nunnenkamp, and C. Bruder, Phys. Rev. Lett. **112**, 094102 (2014).
[11] T. E. Lee, C.-K. Chan, and S. Wang, Phys. Rev. E **89**, 022913 (2014).
[12] T. E. Lee and H. R. Sadeghpour, Phys. Rev. Lett. **111**, 234101 (2013).
[13] T. Weiss, S. Walter, and F. Marquardt, Phys. Rev. A **95**, 041802 (2017).
[14] W. Stefan, N. Andreas, and B. Christoph, Ann. Phys. **527**, 131 (2015).
[15] G. M. Xue, M. Gong, H. K. Xu, W. Y. Liu, H. Deng, Y. Tian, H. F. Yu, Y. Yu, D. N. Zheng, S. P. Zhao, and S. Han, Phys. Rev. B **90**, 224505 (2014).
[16] S. E. Nigg, Phys. Rev. A **97**, 013811 (2018).
[17] M. H. Matheny, M. Grau, L. G. Villanueva, R. B. Karabalin, M. C. Cross, and M. L. Roukes, Phys. Rev. Lett. **112**, 014101 (2014).
[18] W. Li, C. Li, and H. Song, Phys. Rev. E **93**, 062221 (2016).
[19] W. Li, C. Li, and H. Song, Phys. Rev. E **95**, 022204 (2017).
[20] W. Li, C. Li, and H. Song, Phys. Rev. A **95**, 023827 (2017).
[21] W. Li, W. Zhang, C. Li, and H. Song, Phys. Rev. E **96**, 012211 (2017).
[22] A. Mari, A. Farace, N. Didier, V. Giovannetti, and R. Fazio, Phys. Rev. Lett. **111**, 103605 (2013).
[23] W. Li, C. Li, and H. Song, J. Phys. B **48**, 035503 (2015).
[24] F. Bemani, A. Motazedifard, R. Roknizadeh, M. H. Naderi, and D. Vitali, Phys. Rev. A **96**, 023805 (2017).
[25] X.-F. Yin, W.-Z. Zhang, and L. Zhou, J. Mod. Opt. **64**, 578 (2017).
[26] N. Yang, A. Miranowicz, Y. C. Liu, K. Xia, and F. Nori, arXiv:1802.03282.
[27] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[28] B. van der Pol and J. van der Mark, Nature (London) **120**, 363 (1927).
[29] G. L. Giorgi, F. Galve, and R. Zambrini, Phys. Rev. A **94**, 052121 (2016).
[30] T. Weiss, A. Kronwald, and F. Marquardt, New J. Phys. **18**, 013043 (2016).
[31] N. Lörch, S. E. Nigg, A. Nunnenkamp, R. P. Tiwari, and C. Bruder, Phys. Rev. Lett. **118**, 243602 (2017).
[32] F. Galve, G. L. Giorgi, and R. Zambrini, in *Lectures on General Quantum Correlations and their Applications*, edited by F. F. Fanchini, D. O. S. Pinto, and G. Adesso, Quantum Science and Technology Series (Springer, Berlin, 2017), pp. 393–420.
[33] D. Witthaut, S. Wimberger, R. Burioni, and M. Timme, Nat. Commun. **8**, 14829 (2017).
[34] G. L. Giorgi, F. Galve, G. Manzano, P. Colet, and R. Zambrini, Phys. Rev. A **85**, 052101 (2012).
[35] A. Roulet and C. Bruder, Phys. Rev. Lett. **121**, 063601 (2018).
[36] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).
[37] A. Mari and J. Eisert, Phys. Rev. Lett. **103**, 213603 (2009).
[38] A. Mari and J. Eisert, New J. Phys. **14**, 075014 (2012).
[39] H. Tan, G. Li, and P. Meystre, Phys. Rev. A **87**, 033829 (2013).
[40] Y.-D. Wang and A. A. Clerk, Phys. Rev. Lett. **110**, 253601 (2013).
[41] M. J. Woolley and A. A. Clerk, Phys. Rev. A **89**, 063805 (2014).
[42] R.-X. Chen, L.-T. Shen, Z.-B. Yang, H.-Z. Wu, and S.-B. Zheng, Phys. Rev. A **89**, 023843 (2014).
[43] K. Qu and G. S. Agarwal, New J. Phys. **16**, 113004 (2014).
[44] M. Abdi and M. J. Hartmann, New J. Phys. **17**, 013056 (2015).
[45] Z. Li, S.-l. Ma, and F.-l. Li, Phys. Rev. A **92**, 023856 (2015).
[46] Y.-D. Wang, S. Chesi, and A. A. Clerk, Phys. Rev. A **91**, 013807 (2015).
[47] R.-X. Chen, L.-T. Shen, and S.-B. Zheng, Phys. Rev. A **91**, 022326 (2015).
[48] C.-J. Yang, J.-H. An, W. Yang, and Y. Li, Phys. Rev. A **92**, 062311 (2015).
[49] M. Wang, X.-Y. Lü, Y.-D. Wang, J. Q. You, and Y. Wu, Phys. Rev. A **94**, 053807 (2016).
[50] R. X. Chen, C. G. Liao, and X. M. Lin, Sci. Rep. **7**, 14497 (2017).
[51] S. Chakraborty and A. K. Sarma, Phys. Rev. A **97**, 022336 (2018).
[52] C.-G. Liao, R.-X. Chen, H. Xie, and X.-M. Lin, Phys. Rev. A **97**, 042314 (2018).
[53] C.-G. Liao, H. Xie, X. Shang, Z.-H. Chen, and X.-M. Lin, Opt. Express **26**, 13783 (2018).
[54] R. Ralf, W. Andreas, M. Igor, L. Clemens, A. Markus, H. Sungkun, and G. Simon, Nature (London) **556**, 473 (2018).
[55] C. F. Ockeloen-Korppi, E. Damskägg, J.-M. Pirkkalainen, M. Asjad, A. A. Clerk, F. Massel, M. J. Woolley, and M. A. Sillanpää, Nature (London) **556**, 478 (2018).
[56] L. Du, C. H. Fan, H. X. Zhang, and J. H. Wu, Sci. Rep. **7**, 15834 (2017).
[57] H. Geng, L. Du, H. D. Liu, and X. X. Yi, J. Phys. Commun. **2**, 025032 (2018).
[58] S. Sonar, M. Hajdušek, M. Mukherjee, R. Fazio, V. Vedral, S. Vinjanampathy, and L.-C. Kwek, Phys. Rev. Lett. **120**, 163601 (2018).
[59] C. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics* (Springer Science & Business Media, New York, 2004), Vol. 56.
[60] R. Benguria and M. Kac, Phys. Rev. Lett. **46**, 1 (1981).

[61] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).

[62] G. Teschl, *Ordinary Differential Equations and Dynamical Systems* (American Mathematical Society, Providence, RI, 2012), Vol. 140.

[63] E. X. DeJesus and C. Kaufman, Phys. Rev. A **35**, 5288 (1987).

[64] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).

[65] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).